FREE BASIS CONSISTING OF STRICTLY SMALL SECTIONS

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ABSTRACT. Let X be a projective arithmetic variety and $\overline{L}$ a continuous hermitian invertible sheaf on X. In this paper, we would like to consider a mild and appropriate condition to guarantee that $H^0(X, nL)$ has a free basis consisting of strictly small sections for $n \gg 1$.

INTRODUCTION

Let X be a d-dimensional projective arithmetic variety and L an invertible sheaf on X. We fix a continuous hermitian metric $|\cdot|$ of L. In Arakelov geometry, we frequently ask whether $H^0(X, L)$ has a free basis consisting of strictly small sections, that is, sections whose supremum norm are less than 1. However, we know few about this problem in general. For example, Zhang [7] proves that $H^0(X, nL)$ possesses a free basis as above for $n \gg 1$ if the following conditions are satisfied:

(1) $X_{\mathbb{Q}}$ is regular, $L_{\mathbb{Q}}$ is ample and L is nef on every fiber of $X \to \operatorname{Spec}(\mathbb{Z})$.
(2) The metric $|\cdot|$ is C^∞ and the first Chern form $c_1(L, |\cdot|)$ is semipositive.
(3) For some positive integer n_0, there are strictly small sections $s_1, \ldots, s_l$ of $n_0 L$ such that $\{x \in X_{\mathbb{Q}} \mid s_1(x) = \cdots = s_l(x) = 0\} = \emptyset$.

From viewpoint of birational geometry, the ampleness of $L_{\mathbb{Q}}$ is rather strong. Through Example 3.6 and Example 3.7, the base point freeness is not a necessarily condition, but we can realize that the condition (3) is substantially crucial to find a basis consisting of strictly small sections. In this paper, we would like to consider the problem under the mild and appropriate assumption (3) (cf. Corollary B).

Let R be a graded subring of $\bigoplus_{n=0}^{\infty} H^0(X, nL)$ over $\mathbb{Z}$. For each n, we assign a norm $\|\cdot\|_n$ to $R_n \otimes_{\mathbb{Z}} \mathbb{R}$ in such a way that

$$\|s \cdot s'\|_{n+n'} \leq \|s\|_n \cdot \|s'\|_{n'}$$

holds for all $s \in R_n \otimes_{\mathbb{Z}} \mathbb{R}$ and $s' \in R_{n'} \otimes_{\mathbb{Z}} \mathbb{R}$. Then

$$(R, \|\cdot\|) = \bigoplus_{n=0}^{\infty} (R_n, \|\cdot\|_n)$$

is called a *normed graded subring of* L. An important point is that each norm $\|\cdot\|_n$ does not necessarily arise from the metric of L, so that we can obtain several advantages to proceed with arguments. The following theorem is one of the main results of this paper.

Theorem A. *If $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is noetherian and there are homogeneous elements $s_1, \ldots, s_l \in R$ of positive degree such that $\{x \in X_{\mathbb{Q}} \mid s_1(x) = \cdots = s_l(x) = 0\} = \emptyset$, then there is a positive constant B such that*

$$\lambda_{\mathbb{Z}}(R_n, \|\cdot\|_n) \leq B n^{(d+2)(d-1)/2} \left(\max\left\{ \|s_1\|^{1/\deg(s_1)}, \ldots, \|s_l\|^{1/\deg(s_l)} \right\} \right)^n$$

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for all $n \geq 1$, where $\lambda_{\mathbb{Z}}(R_n, \|\cdot\|_n)$ is the infimum of the set of real numbers λ such that there is a free $\mathbb{Z}$-basis $x_1, \ldots, x_r$ of R_n with $\max\{\|x_1\|_n, \ldots, \|x_r\|_n\} \leq \lambda$.

This is a consequence of the technical result Theorem 3.1, which also yields variants of arithmetic Nakai-Moishezon's criterion (cf. Theorem 4.1 and Theorem 4.2). As a corollary of the above theorem, we have the following:

Corollary B. *Let $\overline{L}$ be a continuous hermitian invertible sheaf on X. If the above condition* (3) *is satisfied, in other words,*

$$\left\langle \{s \in H^0(X, n_0 L) \mid \|s\|_{\sup} < 1\} \right\rangle_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_X \to n_0 L$$

is surjective on $X_{\mathbb{Q}}$ for some positive integer n_0, then $H^0(X, nL)$ has a free $\mathbb{Z}$-basis consisting of strictly small sections for $n \geq 1$.

1. Normed $\mathbb{Z}$-module

Let $(V, \|\cdot\|)$ be a normed finite dimensional vector space over $\mathbb{R}$, that is, V is a finite dimensional vector space over $\mathbb{R}$ and $\|\cdot\|$ is a norm of V. Let $\alpha : W \to V$ be an injective homomorphism of finite dimensional vector spaces over $\mathbb{R}$. If we set $\|w\|_{W \hookrightarrow V} = \|\alpha(w)\|$ for $w \in W$, then $\|\cdot\|_{W \hookrightarrow V}$ gives rise to a norm of W. This is called the *subnorm* of W induced by $W \hookrightarrow V$ and the norm $\|\cdot\|$ of V. Next let $\beta : V \to T$ be a surjective homomorphism of finite dimensional vector spaces over $\mathbb{R}$. The *quotient norm* $\|\cdot\|_{V \twoheadrightarrow T}$ of T induced by $V \twoheadrightarrow T$ and the norm $\|\cdot\|$ of V is given by

$$\|t\|_{V \twoheadrightarrow T} = \inf\{\|v\| \mid \beta(v) = t\}$$

for $t \in T$. Let $(U, \|\cdot\|)$ be another normed finite dimensional vector space over $\mathbb{R}$, and let $\phi : V \to U$ be a homomorphism over $\mathbb{R}$. The norm $\|\phi\|$ of ϕ is defined to be

$$\|\phi\| = \sup\{\|\phi(v)\| \mid v \in V, \ \|v\| = 1\}.$$

First let us see the following lemma.

Lemma 1.1. *Let $(V, \|\cdot\|)$ be a normed finite dimensional vector space over $\mathbb{R}$. Let $T \subseteq U \subseteq W \subseteq V$ be vector subspaces of V. Then*

$$(\|\cdot\|_{W \hookrightarrow V})_{W \twoheadrightarrow W/U} = ((\|\cdot\|_{V \twoheadrightarrow V/T})_{W/T \hookrightarrow V/T})_{W/T \twoheadrightarrow W/U}$$

holds on W/U.

Proof. Let us consider the following commutative diagram:

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, two heads] & V \arrow[d, two heads] \\ W/U \arrow[r, hook] & V/U. \end{tikzcd}$$

Then, by [3, (2) in Lemma 3.4], we have

$$(\|\cdot\|_{W \hookrightarrow V})_{W \twoheadrightarrow W/U} = (\|\cdot\|_{V \twoheadrightarrow V/U})_{W/U \hookrightarrow V/U}.$$

Moreover, considering the following commutative diagram:

$$\begin{tikzcd} W/T \arrow[r, hook] \arrow[d, two heads] & V/T \arrow[d, two heads] \\ W/U \arrow[r, hook] & V/U, \end{tikzcd}$$

if we set $\|\cdot\|' = \|\cdot\|_{V \twoheadrightarrow V/T}$, then

$$(\|\cdot\|'_{V/T \twoheadrightarrow V/U})_{W/U \hookrightarrow V/U} = (\|\cdot\|'_{W/T \hookrightarrow V/T})_{W/T \twoheadrightarrow W/U}.$$

Thus the lemma follows because $\|\cdot\|_{V \twoheadrightarrow V/U} = \|\cdot\|'_{V/T \twoheadrightarrow V/U}$ by [3, (1) in Lemma 3.4]. ■

Let M be a finitely generated $\mathbb{Z}$-module and $\|\cdot\|$ a norm of $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. A pair $(M, \|\cdot\|)$ is called a *normed $\mathbb{Z}$-module*. For a normed $\mathbb{Z}$-module $(M, \|\cdot\|)$, we define $\lambda_{\mathbb{Q}}(M, \|\cdot\|)$ and $\lambda_{\mathbb{Z}}(M, \|\cdot\|)$ to be

$$\lambda_{\mathbb{Q}}(M, \|\cdot\|) := \inf \left\{ \lambda \in \mathbb{R} \;\middle|\; \begin{array}{l} \text{there are } e_1, \ldots, e_n \in M \text{ such that } e_1, \ldots, e_n \\ \text{form a basis over } \mathbb{Q} \text{ and } \|e_i\| \leq \lambda \text{ for all } i \end{array} \right\}$$

and

$$\lambda_{\mathbb{Z}}(M, \|\cdot\|) := \inf \left\{ \lambda \in \mathbb{R} \;\middle|\; \begin{array}{l} \text{there are } e_1, \ldots, e_n \in M \text{ such that } e_1, \ldots, e_n \text{ form} \\ \text{a free } \mathbb{Z}\text{-basis of } M/M_{tor} \text{ and } \|e_i\| \leq \lambda \text{ for all } i \end{array} \right\}.$$

Note that if M is a torsion module, then $\lambda_{\mathbb{Q}}(M, \|\cdot\|) = \lambda_{\mathbb{Z}}(M, \|\cdot\|) = 0$.

Lemma 1.2. $\lambda_{\mathbb{Q}}(M, \|\cdot\|) \leq \lambda_{\mathbb{Z}}(M, \|\cdot\|) \leq \operatorname{rk}(M)\lambda_{\mathbb{Q}}(M, \|\cdot\|)$.

Proof. See [6, Lemma 1.7 and its consequence]. ■

Lemma 1.3. *Let $(M_1, \|\cdot\|_1)$ and $(M_2, \|\cdot\|_2)$ be normed $\mathbb{Z}$-modules, and let $\phi : M_1 \to M_2$ be a homomorphism such that ϕ yields an isomorphism over $\mathbb{Q}$. Then we have the following:*

(1) $\lambda_{\mathbb{Q}}(M_2, \|\cdot\|_2) \leq \|\phi\| \lambda_{\mathbb{Q}}(M_1, \|\cdot\|_1)$.
(2) *Further we assume that ϕ is surjective and that ϕ induces an isometry*

$$((M_1)_{\mathbb{R}}, \|\cdot\|_1) \xrightarrow{\sim} ((M_2)_{\mathbb{R}}, \|\cdot\|_2).$$

Then $\lambda_{\mathbb{Q}}(M_2, \|\cdot\|_2) = \lambda_{\mathbb{Q}}(M_1, \|\cdot\|_1)$.

Proof. (1) Let $e_1, \ldots, e_n \in M_1$ such that $e_1, \ldots, e_n$ form a basis of M_1 over $\mathbb{Q}$ and

$$\max\{\|e_1\|_1, \ldots, \|e_n\|_1\} = \lambda_{\mathbb{Q}}(M_1, \|\cdot\|_1).$$

Then $\phi(e_1), \ldots, \phi(e_n)$ form a basis of M_2 over $\mathbb{Q}$ and

$$\|\phi(e_i)\|_2 \leq \|\phi\| \|e_i\|_1 \leq \|\phi\| \lambda_{\mathbb{Q}}(M_1, \|\cdot\|_1)$$

for all i. Thus we have the assertion.

(2) First of all, by (1), $\lambda_{\mathbb{Q}}(M_2, \|\cdot\|_2) \leq \lambda_{\mathbb{Q}}(M_1, \|\cdot\|_1)$. Let $y_1, \ldots, y_n \in M_2$ such that $y_1, \ldots, y_n$ form a basis of M_2 over $\mathbb{Q}$ and

$$\max\{\|y_1\|_2, \ldots, \|y_n\|_2\} = \lambda_{\mathbb{Q}}(M_2, \|\cdot\|_2).$$

For each i, we choose $x_i \in M_1$ with $\phi(x_i) = y_i$. Then $\|x_i\|_1 = \|y_i\|_2$ for all i. Thus $\lambda_{\mathbb{Q}}(M_1, \|\cdot\|_1) \leq \lambda_{\mathbb{Q}}(M_2, \|\cdot\|_2)$. ■

Proposition 1.4. *Let $(M_1, \|\cdot\|_1), \ldots, (M_n, \|\cdot\|_n)$ be normed $\mathbb{Z}$-modules. For each i with $2 \leq i \leq n$, let $\alpha_i : M_{i-1} \to M_i$ be a homomorphism such that α_i gives rise to an injective homomorphism over $\mathbb{Q}$. We set*

$$\phi_i = \alpha_n \circ \cdots \circ \alpha_{i+1} : M_i \to M_n$$

for $i = i, \ldots, n-1$, and

$$Q_i = \begin{cases} \operatorname{Coker}(\alpha_i : M_{i-1} \to M_i) & \text{if } i \geq 2, \\ M_1 & \text{if } i = 1 \end{cases}$$

for $i = 1, \ldots, n$. Then we have

$$\lambda_{\mathbb{Q}}(M_n, \|\cdot\|_n) \leq \lambda_{\mathbb{Q}}(Q_n, \|\cdot\|_{n, M_n \twoheadrightarrow Q_n}) + \sum_{i=1}^{n-1} \|\phi_i\| \lambda_{\mathbb{Q}}(Q_i, \|\cdot\|_{i, M_i \twoheadrightarrow Q_i}) \operatorname{rk} Q_i.$$

Proof. The proof of this proposition can be found in [6, Lemma 5.1]. For reader's convenience, we reprove it here.

Let $\|\cdot\|_i' = \|\cdot\|_{n,(M_i)_{\mathbb{R}} \hookrightarrow (M_n)_{\mathbb{R}}}$, that is, the sub-norm induced by the injective homomorphism $\phi_i : (M_i)_{\mathbb{R}} \to (M_n)_{\mathbb{R}}$ and the norm $\|\cdot\|_n$ of $(M_n)_{\mathbb{R}}$. First let us see the following claim.

Claim 1.4.1. $\lambda_{\mathbb{Q}}(Q_i, \|\cdot\|'_{i, M_i \twoheadrightarrow Q_i}) \leq \|\phi_i\| \lambda_{\mathbb{Q}}(Q_i, \|\cdot\|_{i, M_i \twoheadrightarrow Q_i})$.

By the definition of $\|\phi_i\|$, for $x \in (M_i)_{\mathbb{R}}$,

$$\|x\|_i \|\phi_i\| \geq \|\phi_i(x)\| = \|x\|_i'.$$

Thus, for $y \in (Q_i)_{\mathbb{R}}$,

$$\|y\|_{i, M_i \twoheadrightarrow Q_i} \|\phi_i\| \geq \|y\|'_{i, M_i \twoheadrightarrow Q_i},$$

which shows the inequality of the claim. □

By Claim 1.4.1, (2) in Lemma 1.3 and replacing M_i with $\phi_i(M_i)$, we may assume that $\alpha_i : M_{i-1} \hookrightarrow M_i$ is an inclusion map and $\|\cdot\|_i = \|\cdot\|_{n, M_i \hookrightarrow M_n}$.

Claim 1.4.2. *The assertion holds in the case $n = 2$, that is,*

$$\lambda_{\mathbb{Q}}(M_2, \|\cdot\|_2) \leq \lambda_{\mathbb{Q}}(Q_2, \|\cdot\|_{2, M_2 \twoheadrightarrow Q_2}) + \lambda_{\mathbb{Q}}(M_1, \|\cdot\|_1) \operatorname{rk} M_1.$$

Let $e_1, \ldots, e_s \in M_1$ and $f_1, \ldots, f_t \in Q_2$ such that $e_1, \ldots, e_s$ and $f_1, \ldots, f_t$ form bases of M_1 and Q_2 over $\mathbb{Q}$ respectively, and that

$$\begin{cases} \lambda_{\mathbb{Q}}(M_1, \|\cdot\|_1) = \max\{\|e_1\|_1, \ldots, \|e_s\|_1\}, \\ \lambda_{\mathbb{Q}}(Q_2, \|\cdot\|_{2, M_2 \twoheadrightarrow Q_2}) = \max\{\|f_1\|_{2, M_2 \twoheadrightarrow Q_2}, \ldots, \|f_t\|_{2, M_2 \twoheadrightarrow Q_2}\}. \end{cases}$$

Let us choose $f_j' \in M_2$ and $f_j'' \in (M_2)_{\mathbb{R}}$ such that $f_j' = f_j$ on Q_2, $f_j'' = f_j$ on $(Q_2)_{\mathbb{R}}$ and that $\|f_j''\|_2 = \|f_j\|_{2, M_2 \twoheadrightarrow Q_2}$. Since $f_j' \otimes 1 - f_j'' \in (M_1)_{\mathbb{R}}$, there are $a_{ji} \in \mathbb{R}$ such that

$$f_j' \otimes 1 - f_j'' = \sum_i a_{ji}(e_i \otimes 1).$$

We set $g_j = f_j' - \sum_i \lfloor a_{ji} \rfloor e_i$. Then $e_1, \ldots, e_s, g_1, \ldots, g_t \in M_2$ form a basis of M_2 over $\mathbb{Q}$. Moreover, as

$$g_j \otimes 1 = f_j'' + \sum_i (a_{ji} - \lfloor a_{ji} \rfloor)(e_i \otimes 1),$$

we have

$$\|g_j\|_2 \leq \lambda_{\mathbb{Q}}(Q_2, \|\cdot\|_{2, M_2 \twoheadrightarrow Q_2}) + \lambda_{\mathbb{Q}}(M_1, \|\cdot\|_1) \operatorname{rk} M_1,$$

which implies the claim. □

We assume $n \geq 3$. We set $M_i' = M_i / M_1$ for $i = 2, \ldots, n$ and the norm $\|\cdot\|_i'$ of M_i' is given by $\|\cdot\|_i' = \|\cdot\|_{i, M_i \twoheadrightarrow M_i'}$. Note that

$$\|\cdot\|_i' = (\|\cdot\|_{n, M_n \twoheadrightarrow M_n'})_{M_i' \hookrightarrow M_n'}$$

by [3, (2) in Lemma 3.4]. Applying the induction hypothesis to

$$(M_2', \|\cdot\|_2') \hookrightarrow \cdots \hookrightarrow (M_n', \|\cdot\|_n'),$$

we obtain

$$\lambda_{\mathbb{Q}}(M'_n, \|\cdot\|'_n) \leq \lambda_{\mathbb{Q}}(Q_n, \|\cdot\|'_{n,M'_n \twoheadrightarrow Q_n}) + \sum_{i=2}^{n-1} \lambda_{\mathbb{Q}}(Q_i, \|\cdot\|'_{i,M'_i \twoheadrightarrow Q_i}) \operatorname{rk} Q_i.$$

Using Lemma 1.1 in the case where

$$M_1 \subseteq M_{i-1} \subseteq M_i \subseteq M_n,$$

we have $\|\cdot\|'_{i,M'_i \twoheadrightarrow Q_i} = \|\cdot\|_{i,M_i \twoheadrightarrow Q_i}$. Therefore, the above inequality means

$$\lambda_{\mathbb{Q}}(M'_n, \|\cdot\|'_n) \leq \lambda_{\mathbb{Q}}(Q_n, \|\cdot\|_{n,M_n \twoheadrightarrow Q_n}) + \sum_{i=2}^{n-1} \lambda_{\mathbb{Q}}(Q_i, \|\cdot\|_{i,M_i \twoheadrightarrow Q_i}) \operatorname{rk} Q_i. \tag{1.4.3}$$

On the other hand, applying Claim 1.4.2 to the case where $(M_1, \|\cdot\|_1) \hookrightarrow (M_n, \|\cdot\|_n)$, we can see

$$\lambda_{\mathbb{Q}}(M_n, \|\cdot\|_n) \leq \lambda_{\mathbb{Q}}(M'_n, \|\cdot\|'_n) + \lambda_{\mathbb{Q}}(M_1, \|\cdot\|_1) \operatorname{rk} M_1, \tag{1.4.4}$$

so that we obtain the assertion combing (1.4.3) with (1.4.4). ■

2. Normed graded ring

Let k be a commutative ring with unity and $R = \bigoplus_{n=0}^{\infty} R_n$ a graded ring over k. Let M be a R-module and h a positive integer. We say M is a *h-graded R-module* if M has a decomposition $M = \bigoplus_{n=-\infty}^{\infty} M_n$ as k-modules and

$$x \in R_n,\ m \in M_{n'} \quad \Longrightarrow \quad x \cdot m \in M_{hn+n'}$$

holds for all $n \in \mathbb{Z}_{\geq 0}$ and $n' \in \mathbb{Z}$. For example, if we set $R^{(h)} = \bigoplus_{n=0}^{\infty} R_{nh}$, then R is a h-graded $R^{(h)}$-module. Form now on, we assume that $k = \mathbb{Z}$ and R_n (resp. M_n) is a finitely generated $\mathbb{Z}$-module for all $n \in \mathbb{Z}_{\geq 0}$ (resp. $n \in \mathbb{Z}$). Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. We set $R_{\mathbb{K}} = R \otimes_{\mathbb{Z}} \mathbb{K}$ and $M_{\mathbb{K}} = M \otimes_{\mathbb{Z}} \mathbb{K}$. Then

$$R_{\mathbb{K}} = \bigoplus_{n=0}^{\infty} (R_n)_{\mathbb{K}} \quad \text{and} \quad M_{\mathbb{K}} = \bigoplus_{n=-\infty}^{\infty} (M_n)_{\mathbb{K}},$$

where $(R_n)_{\mathbb{K}} = R_n \otimes_{\mathbb{Z}} \mathbb{K}$ and $(M_n)_{\mathbb{K}} = M_n \otimes_{\mathbb{Z}} \mathbb{K}$. Note that $R_{\mathbb{K}}$ is a graded ring over $\mathbb{K}$ and $M_{\mathbb{K}}$ is a h-graded $R_{\mathbb{K}}$-module. We say

$$(R, \|\cdot\|) = \bigoplus_{n=0}^{\infty} (R_n, \|\cdot\|_n)$$

is a *normed graded ring over* $\mathbb{Z}$ if

(1) $\|\cdot\|_n$ is a norm of $(R_n)_{\mathbb{R}}$ for each $n \in \mathbb{Z}_{\geq 0}$, and
(2) $\|s \cdot s'\|_{n+n'} \leq \|s\|_n \|s'\|_{n'}$ holds for all $s \in (R_n)_{\mathbb{R}}$ and $s' \in (R_{n'})_{\mathbb{R}}$.

Similarly,

$$(M, \|\cdot\|_M) = \bigoplus_{n=-\infty}^{\infty} (M_n, \|\cdot\|_{M_n})$$

is called a *normed h-graded $(R, \|\cdot\|)$-module* if

(1)' $\|\cdot\|_{M_n}$ is a norm of $(M_n)_{\mathbb{R}}$ for each $n \in \mathbb{Z}$, and
(2)' $\|s \cdot m\|_{M_{hn+n'}} \leq \|s\|_n \|m\|_{M_{n'}}$ holds for all $s \in (R_n)_{\mathbb{R}}$ and $m \in (M_{n'})_{\mathbb{R}}$.

Proposition 2.1. *Let I be a homogeneous ideal of R and $R' = R/I$. Let $f : M \to Q$ be a surjective homomorphism of h-graded R-modules of degree 0, that is, $f(M_n) = Q_n$ for all $n \in \mathbb{Z}$. We set*

$$(R', \|\cdot\|') = \bigoplus_{n=0}^{\infty} (R'_n, \|\cdot\|'_n) \quad \textit{and} \quad (Q, \|\cdot\|_Q) = \bigoplus_{n=-\infty}^{\infty} (Q_n, \|\cdot\|_{Q_n}),$$

where $\|\cdot\|'_n = \|\cdot\|_{n, R_n \twoheadrightarrow R'_n}$ and $\|\cdot\|_{Q_n} = \|\cdot\|_{M_n, M_n \twoheadrightarrow Q_n}$. Then we have the following:

(1) *$(R', \|\cdot\|')$ is a normed graded ring over $\mathbb{Z}$.*
(2) *If $I \cdot Q = 0$, then $(Q, \|\cdot\|_Q)$ is naturally a normed h-graded $(R', \|\cdot\|')$-module.*

Proof. (1) We need to see that

$$\|x' \cdot y'\|'_{n+n'} \le \|x'\|'_n \|y'\|'_{n'}$$

for all $x' \in (R'_n)_{\mathbb{R}}$ and $y' \in (R'_{n'})_{\mathbb{R}}$. Indeed, we choose $x \in (R_n)_{\mathbb{R}}$ and $y \in (R_{n'})_{\mathbb{R}}$ such that the classes of x and y in $R'_{\mathbb{R}}$ are x' and y' respectively and that $\|x\|_n = \|x'\|'_n$ and $\|y\|_n = \|y'\|'_{n'}$. Then, as the class of $x \cdot y$ in $R'_{\mathbb{R}}$ is $x' \cdot y'$,

$$\|x' \cdot y'\|'_{n+n'} \le \|x \cdot y\|_{n+n'} \le \|x\|_n \|y\|_{n'} = \|x'\|'_n \|y'\|'_{n'}.$$

(2) It is sufficient to show that

$$\|x' \cdot q\|_{Q_{hn+n'}} \le \|x'\|'_n \|q\|_{Q_{n'}}$$

for all $x' \in (R'_n)_{\mathbb{R}}$ and $q \in (Q_{n'})_{\mathbb{R}}$, which can be checked in the same way as in (1). ■

Next let us observe the following lemma:

Lemma 2.2. *We assume the following:*

(1) *$M_{\mathbb{Q}}$ is a finitely generated $R_{\mathbb{Q}}$-module, and $M_n = \{0\}$ for $n < 0$.*
(2) *There are $A, e, \upsilon \in \mathbb{R}_{>0}$ such that $\lambda_{\mathbb{Q}}(R_n, \|\cdot\|_n) \le A n^e \upsilon^n$ for all $n \ge 1$.*

Then there is $A' \in \mathbb{R}_{>0}$ such that $\lambda_{\mathbb{Q}}(M_n, \|\cdot\|_{M_n}) \le A' n^e \upsilon^{n/h}$ for all $n \ge 1$.

Proof. For $n \ge 1$, we choose $s_{n,1}, \ldots, s_{n,r_n} \in R_n$ such that $s_{n,1}, \ldots, s_{n,r_n}$ form a basis of $(R_n)_{\mathbb{Q}}$ and $\|s_{n,j}\|_n \le A n^e \upsilon^n$ holds for all $j = 1, \ldots, r_n$. Let $m_1, \ldots, m_l$ be homogeneous elements of $M_{\mathbb{Q}}$ such that $M_{\mathbb{Q}}$ is generated by $m_1, \ldots, m_l$ as a $R_{\mathbb{Q}}$-module. Let a_i be the degree of m_i. Clearly we may assume that $m_i \in M_{a_i}$ by replacing m_i with $b m_i$ ($b \in \mathbb{Z}_{>0}$). If $n > \max\{a_1, \ldots, a_l\}$, then $(M_n)_{\mathbb{Q}}$ is generated by elements of the form $s_{i,j} m_k$ with $ih + a_k = n$ and $i \ge 1$. We set

$$B = \max_{k=1,\ldots,l} \left\{ \frac{\|m_k\|_{M_{a_k}} \upsilon^{-a_k/h}}{h^e} \right\}.$$

Note that $s_{i,j} m_k \in M_n$ and

$$\begin{aligned} \|s_{i,j} m_k\|_{M_n} &\le \|s_{i,j}\|_i \|m_k\|_{M_{a_k}} \le A i^e \upsilon^i \|m_k\|_{M_{a_k}} \\ &= A \left(\frac{n - a_k}{h} \right)^e \upsilon^{(n-a_k)/h} \|m_k\|_{M_{a_k}} \le A B n^e \upsilon^{n/h} \end{aligned}$$

which means that $\lambda_{\mathbb{Q}}(M_n, \|\cdot\|_{M_n}) \le A B n^e \upsilon^{n/h}$ holds for all $n > \max\{a_1, \ldots, a_l\}$, as required. ■

As a consequence, we have the following proposition.

Proposition 2.3. *Let I, J and K be homogeneous ideals of R such that $J \subseteq K$ and $I \cdot K \subseteq J$. We set $R' = R/I$ as before and $Q = K/J$. Let $\|\cdot\|_{K_n} = \|\cdot\|_{n, K_n \hookrightarrow R_n}$ and $\|\cdot\|_{Q_n} = \|\cdot\|_{K_n, K_n \twoheadrightarrow Q_n}$. If $R_{\mathbb{Q}}$ is noetherian and there are $A, e, v \in \mathbb{R}_{>0}$ such that*

$$\lambda_{\mathbb{Q}}(R'_n, \|\cdot\|'_n) \leq A n^e v^n$$

for all $n \geq 1$, then there is $A' \in \mathbb{R}_{>0}$ such that

$$\lambda_{\mathbb{Q}}(Q_n, \|\cdot\|_{Q_n}) \leq A' n^e v^n$$

for all $n \geq 1$.

Proof. Obviously, $(K, \|\cdot\|_K) = \bigoplus_{n=0}^{\infty} (K_n, \|\cdot\|_{K_n})$ is a normed 1-graded $(R, \|\cdot\|)$-module. Thus, by Proposition 2.1, $(Q, \|\cdot\|_Q) = \bigoplus_{n=0}^{\infty} (Q_n, \|\cdot\|_{Q_n})$ is also a normed 1-graded $(R, \|\cdot\|)$-module. As $I \cdot Q = 0$, by Proposition 2.1 again, $(Q, \|\cdot\|_Q)$ is a normed 1-graded $(R', \|\cdot\|')$-module. Since $R_{\mathbb{Q}}$ is noetherian and $K_{\mathbb{Q}}$ is an ideal of $R_{\mathbb{Q}}$, $K_{\mathbb{Q}}$ is finitely generated as a $R_{\mathbb{Q}}$-module. Thus $Q_{\mathbb{Q}}$ is also finitely generated as a $R'_{\mathbb{Q}}$-module. Hence the assertion follows from Lemma 2.2. ■

Finally note the following lemma, which will be used later.

Lemma 2.4. *Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a graded ring and h a positive integer. If R is noetherian, then $R^{(h)}$ is also noetherian and R is a finitely generated $R^{(h)}$-module.*

Proof. See [1, Chap. III, § 1, n° 3, Proposition 2 and its proof]. ■

3. Estimation of $\lambda_{\mathbb{Q}}$ for a normed graded ring

Let X be a d-dimensional projective arithmetic variety, that is, X is a d-dimensional projective and flat integral scheme over $\mathbb{Z}$, and let L be an invertible sheaf on X. Let R be a graded subring of $\bigoplus_{n=0}^{\infty} H^0(X, nL)$ over $\mathbb{Z}$. Such a graded ring R is called a *graded subring of* L. For each n, we assign a norm $\|\cdot\|_n$ to $(R_n)_{\mathbb{R}}$ such that $(R, \|\cdot\|) = \bigoplus_{n=0}^{\infty} (R_n, \|\cdot\|_n)$ is a normed graded ring over $\mathbb{Z}$.

For an ideal sheaf $\mathcal{I}$ of X, we set

$$\begin{cases} I_n(R; \mathcal{I}) = H^0(X, nL \otimes \mathcal{I}) \cap R_n, \\ I(R; \mathcal{I}) = \bigoplus_{n=0}^{\infty} I_n(R; \mathcal{I}), \\ R_{\mathcal{I}} = R/I(R; \mathcal{I}). \end{cases}$$

Then $I(R; \mathcal{I})$ is a homogeneous ideal of R. Let $\|\cdot\|_{(R_{\mathcal{I}})_n}$ be the quotient norm of $(R_{\mathcal{I}})_n$ induced by $R_n \twoheadrightarrow (R_{\mathcal{I}})_n$ and the norm $\|\cdot\|_n$ of R_n. Let Y be an arithmetic subvariety of X, that is, Y is an integral closed subscheme flat over $\mathbb{Z}$, and $\mathcal{I}_Y$ the defining ideal sheaf of Y. Then, for simplicity, $R_{\mathcal{I}_Y}$, $\|\cdot\|_{R_{\mathcal{I}_Y}}$, $(R_Y)_n$, and $\|\cdot\|_{(R_Y)_n}$ are denoted by R_Y, $\|\cdot\|_Y$, $R_{Y,n}$ and $\|\cdot\|_{Y,n}$ respectively. Note that

$$R_{Y,n} \hookrightarrow H^0(X, nL)/H^0(X, nL \otimes \mathcal{I}_Y) \hookrightarrow H^0(Y, nL|_Y).$$

Thus R_Y is a graded subring of $L|_Y$ and

$$R_{Y,n} \xrightarrow{\sim} \operatorname{Image}(R_n \to H^0(Y, nL|_Y)).$$

In particular, R_Y is an integral domain. We denote the set of all arithmetic subvarieties of X by Σ_X. The following theorem is the technical main theorem of this paper.

Theorem 3.1. *Let $v : \Sigma_X \to \mathbb{R}_{>0}$ be a map. For $(R, \|\cdot\|)$ and v, we assume the following:*

(1) *$R_{\mathbb{Q}}$ is noetherian.*

(2) *For each* $Y \in \Sigma_X$, *there is* $n_0 \in \mathbb{Z}_{>0}$ *such that* $(R_{Y,n})_{\mathbb{Q}} = H^0\left(Y_{\mathbb{Q}}, nL_{\mathbb{Q}}|_{Y_{\mathbb{Q}}}\right)$ *for all* $n \geq n_0$.

(3) *For each* $Y \in \Sigma_X$, *there are* $n_1 \in \mathbb{Z}_{>0}$ *and* $s \in R_{Y,n_1} \setminus \{0\}$ *with* $\|s\|_{Y,n_1} \leq \upsilon(Y)^{n_1}$.

Then there are $B \in \mathbb{R}_{>0}$ *and a finite subset* S *of* Σ_X *such that*

$$\lambda_{\mathbb{Q}}(R_n, \|\cdot\|_n) \leq Bn^{d(d-1)/2} \left(\max\{\upsilon(Y) \mid Y \in S\}\right)^n$$

for all $n \geq 1$.

Proof. This theorem can be proved by similar techniques as in [7, Theorem (4.2)]. Let $D \in \Sigma_X$ and $\upsilon_D = \upsilon|_{\Sigma_D}$, where Σ_D is the set of all arithmetic subvarieties of D. Note that the conditions (1), (2) and (3) also hold for $(R_D, \|\cdot\|_D)$ and υ_D. Let us begin with the following claim.

Claim 3.1.1. *We may assume that there is a non-zero* $s \in R_1$ *with* $\|s\|_1 \leq \upsilon(X)$.

We choose a positive integer m and a non-zero section $s \in R_m$ with $\|s\|_m \leq \upsilon(X)^m$. Clearly the assumptions (1) and (2) of the theorem hold for $R^{(m)} = \bigoplus_{n=0}^{\infty} R_{mn}$. For $Y \in \Sigma_X$,we choose a positive integer n_1 and a non-zero $t \in R_{Y,n_1}$ with $\|t\|_{Y,n_1} \leq \upsilon(Y)^{n_1}$. Then $t^m \in R_{Y,mn_1} \setminus \{0\}$ and

$$\|t^m\|_{Y,mn_1} \leq (\|t\|_{Y,n_1})^m \leq (\upsilon(Y)^m)^{n_1}.$$

Thus $(R^{(m)}, \|\cdot\|^{(m)})$ and υ^m satisfy the assumption (3) of the theorem. Therefore, if the theorem holds for $(R^{(m)}, \|\cdot\|^{(m)})$ and υ^m, then there are $B \in \mathbb{R}_{>0}$ and a finite subset S of Σ_X such that

$$\lambda_{\mathbb{Q}}(R_{nm}, \|\cdot\|_{nm}) \leq Bn^{d(d-1)/2} \left(\max\{\upsilon(Y)^m \mid Y \in S\}\right)^n$$

for all $n \geq 1$. On the other hand, by Lemma 2.4, $R_{\mathbb{Q}}$ is a finitely generated $R_{\mathbb{Q}}^{(m)}$-module. Thus, by Lemma 2.2, there is $B' \in \mathbb{R}_{>0}$ such that

$$\lambda_{\mathbb{Q}}(R_n, \|\cdot\|_n) \leq B'n^{d(d-1)/2} \left(\max\{\upsilon(Y)^m \mid Y \in S\}\right)^{n/m}$$

for all $n \geq 1$. Therefore the claim follows. □

Claim 3.1.2. *The assertion of the theorem holds if* $d = 1$.

Since $R_n \xrightarrow{\cdot s} R_{n+1}$ is injective,

$$\operatorname{rk} R_1 \leq \cdots \leq \operatorname{rk} R_n \leq \operatorname{rk} R_{n+1} \leq \cdots \leq \operatorname{rk} L.$$

Thus there is a positive integer n_0 such that $R_{n_0} \xrightarrow{\cdot s^n} R_{n_0+n}$ yields an isomorphism over $\mathbb{Q}$. Hence, by (1) in Lemma 1.3,

$$\lambda_{\mathbb{Q}}(R_{n+n_0}, \|\cdot\|_{n+n_0}) \leq \|s\|_1^n \lambda_{\mathbb{Q}}(R_{n_0}, \|\cdot\|_{n_0}) \leq \upsilon(X)^n \lambda_{\mathbb{Q}}(R_{n_0}, \|\cdot\|_{n_0}),$$

as required. □

We prove the theorem on induction of d. By Claim 3.1.2, we have done in the case $d = 1$. Thus we assume $d > 1$. Let $\mathcal{I}$ be the ideal sheaf of $\mathcal{O}_X$ given by

$$\mathcal{I} = \operatorname{Image}\left(L^{-1} \xrightarrow{\otimes s} \mathcal{O}_X\right).$$

Claim 3.1.3. *There is a sequence*

$$\mathcal{I}_0 = \mathcal{I} \subsetneq \mathcal{I}_1 \subsetneq \cdots \subsetneq \mathcal{I}_m = \mathcal{O}_X$$

of ideal sheaves and proper integral subschemes $D_1, \ldots, D_m$ of X such that $\mathcal{I}_{D_r} \cdot \mathcal{I}_r \subseteq \mathcal{I}_{r-1}$ for all $r = 1, \ldots, m$, where $\mathcal{I}_{D_r}$ is the defining ideal sheaf of D_r.

It is standard. For example, we can show it by using [2, Chapter 1, Proposition 7.4]. □

Let us fix a positive integer n_1 such that $(R_n)_{\mathbb{Q}} = H^0(X_{\mathbb{Q}}, nL_{\mathbb{Q}})$ for all $n \geq n_1$. We set

$$\overline{R}_n = (R_n, \|\cdot\|_n) \quad \text{and} \quad \overline{I}_n(R; \mathcal{I}_r) = (I_n(R; \mathcal{I}_r), \|\cdot\|_{n,r}),$$

where $\|\cdot\|_{n,r} = \|\cdot\|_{n, I_n(R;\mathcal{I}_r) \hookrightarrow R_n}$. Note that $\overline{R}_n = \overline{I}_n(R; \mathcal{I}_m)$. We would like to apply Proposition 1.4 to

(3.1.4)

$$\begin{array}{cccccccc}
\overline{R}_{n_1} & \xrightarrow{\cdot s} & \overline{I}_{n_1+1}(R;\mathcal{I}_0) & \hookrightarrow \cdots \hookrightarrow & \overline{I}_{n_1+1}(R;\mathcal{I}_r) & \hookrightarrow \cdots \hookrightarrow & \overline{I}_{n_1+1}(R;\mathcal{I}_m) \\
 & \xrightarrow{\cdot s} & \overline{I}_{n_1+2}(R;\mathcal{I}_0) & \hookrightarrow \cdots \hookrightarrow & \overline{I}_{n_1+2}(R;\mathcal{I}_r) & \hookrightarrow \cdots \hookrightarrow & \overline{I}_{n_1+2}(R;\mathcal{I}_m) \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \xrightarrow{\cdot s} & \overline{I}_{n}(R;\mathcal{I}_0) & \hookrightarrow \cdots \hookrightarrow & \overline{I}_{n}(R;\mathcal{I}_r) & \hookrightarrow \cdots \hookrightarrow & \overline{I}_{n}(R;\mathcal{I}_m).
\end{array}$$

For this purpose, let us observe the following claim.

Claim 3.1.5. (a) *Let $\|\cdot\|_{n,r,\mathrm{quot}}$ be the quotient norm of $I_n(R;\mathcal{I}_r)/I_n(R;\mathcal{I}_{r-1})$ induced by $I_n(R;\mathcal{I}_r) \twoheadrightarrow I_n(R;\mathcal{I}_r)/I_n(R;\mathcal{I}_{r-1})$ and $\|\cdot\|_{n,r}$ of $I_n(R;\mathcal{I}_r)$. Then, for each $1 \leq r \leq m$, there are $B_r \in \mathbb{R}_{>0}$ and a finite subset S_r of Σ_X such that*

$$\lambda_{\mathbb{Q}}\left(I_n(R;\mathcal{I}_r)/I_n(R;\mathcal{I}_{r-1}),\ \|\cdot\|_{n,r,\mathrm{quot}}\right) \leq B_r n^{(d-1)(d-2)/2} \left(\max\{\upsilon(Y) \mid Y \in S_r\}\right)^n.$$

for all $n \geq 1$.

(b) *If we set*

$$e_{n,r} = \max\{1, \mathrm{rk}(I_n(R;\mathcal{I}_r)/I_n(R;\mathcal{I}_{r-1}))\},$$

then there is $C_1 \in \mathbb{R}_{>0}$ such that $e_{n,r} \leq C_1 n^{d-2}$ for all $n \geq 1$ and $r = 1, \ldots, m$.

(c) $\mathrm{rk}(I_n(R;\mathcal{I}_0)/R_{n-1}s) = 0$ *for all $n \geq n_1 + 1$.*

(a) If D_r is vertical, then $I_n(R;\mathcal{I}_r)/I_n(R;\mathcal{I}_{r-1})$ is a torsion module for all $n \geq 0$. Thus the assertion is obvious. In this case, we can set $S_r = \{X\}$ and $B_r = 1$. Otherwise, since $I(R;\mathcal{I}_{D_r}) \cdot I(R;\mathcal{I}_r) \subseteq I(R;\mathcal{I}_{r-1})$, the assertion follows from Proposition 2.3 and the hypothesis of induction.

(b) Note that $I_n(R;\mathcal{I}_r)/I_n(R;\mathcal{I}_{r-1}) \hookrightarrow H^0(D_r, nL \otimes \mathcal{I}_r/\mathcal{I}_{r-1})$.

(c) It follows from

$$(R_{n-1})_{\mathbb{Q}} s = H^0(X_{\mathbb{Q}}, (n-1)L_{\mathbb{Q}})s = H^0(X_{\mathbb{Q}}, (nL \otimes \mathcal{I})_{\mathbb{Q}}) = I_n(R;\mathcal{I})_{\mathbb{Q}}.$$

□

Using (c) in Claim 3.1.5 and applying Proposition 1.4 to (3.1.4), we obtain

$$\begin{aligned}
\lambda_{\mathbb{Q}}(R_n, \|\cdot\|_n) &\leq \sum_{i=n_1+1}^{n} \left(\sum_{r=1}^{m} \|s\|_1^{n-i} \lambda_{\mathbb{Q}}\left(I_i(R;\mathcal{I}_r)/I_i(R;\mathcal{I}_{r-1}),\ \|\cdot\|_{i,r,\mathrm{quot}}\right) e_{i,r} \right) \\
&\qquad + \|s\|_1^{n-n_1} \lambda(R_{n_1}, \|\cdot\|_{n_1}) \, \mathrm{rk}(R_{n_1})
\end{aligned}$$

for $n \geq n_1 + 1$. Hence, if we set $S = S_1 \cup \cdots \cup S_r \cup \{X\}$, then, using (a) and (b) in Claim 3.1.5, the theorem follows. ■

For homogeneous elements $s_1, \ldots, s_l$ of R, we define $\mathrm{Bs}_{\mathbb{Q}}(s_1, \ldots, s_l)$ to be

$$\mathrm{Bs}_{\mathbb{Q}}(s_1, \ldots, s_l) = \{x \in X_{\mathbb{Q}} \mid s_1(x) = \cdots = s_l(x) = 0\}.$$

As an application of Theorem 3.1, we have the following theorem.

Theorem 3.2. *If $R_{\mathbb{Q}}$ is noetherian and there are homogeneous elements $s_1, \ldots, s_l \in R$ of positive degree such that $\mathrm{Bs}_{\mathbb{Q}}(s_1, \ldots, s_l) = \emptyset$, then there is a positive constant B such that*

$$\lambda_{\mathbb{Q}}(R_n, \|\cdot\|_n) \leq B n^{d(d-1)/2} \left(\max \left\{ \|s_1\|^{1/\deg(s_1)}, \ldots, \|s_l\|^{1/\deg(s_l)} \right\} \right)^n,$$

for all $n \geq 1$.

Proof. Let us begin with the following claim:

Claim 3.2.1. *We may assume that R is generated by R_1 over R_0 and that $s_1, \ldots, s_l \in R_1$.*

Since $R_{\mathbb{Q}}$ is noetherian, there are homogeneous elements $x_1, \ldots, x_r \in R_{\mathbb{Q}}$ such that $R_{\mathbb{Q}} = (R_0)_{\mathbb{Q}}[x_1, \ldots, x_r]$ (cf. [1, Chap. III, § 1, n° 2, Corollaire]). Replacing x_i with $m x_i$ ($m \in \mathbb{Z}_{>0}$), we may assume that $x_i \in R$ for all i. We set

$$R' = R_0[x_1, \ldots, x_r, s_1, \ldots, s_l]$$

in R. Then $R'_{\mathbb{Q}} = R_{\mathbb{Q}}$. As R_n/R'_n is a torsion module, by (1) in Lemma 1.3, we have $\lambda_{\mathbb{Q}}(R_n, \|\cdot\|_n) \leq \lambda_{\mathbb{Q}}(R'_n, \|\cdot\|_n)$ for all $n \geq 0$. Thus we may assume that R is noetherian. Therefore, there is a positive integer h such that $R^{(h)}$ is generated by R_h over R_0 (cf. [1, Chap. III, § 1, n° 3, Proposition 3]). Letting a_i be the degree of s_i, we set $a = a_1 \cdots a_l$ and $s'_i = s_i^{h a_1 \cdots a_{i-1} a_{i+1} \cdots a_l}$ for each i. Then $s'_1, \ldots, s'_l \in R_{ah}$ and

$$\max\{\|s'_1\|, \ldots, \|s'_l\|\} \leq \left(\max \left\{ \|s_1\|^{1/\deg(s_1)}, \ldots, \|s_l\|^{1/\deg(s_l)} \right\} \right)^{ah}.$$

Moreover, $R^{(ah)}$ is generated by R_{ah} over R_0. Thus, as in Claim 3.1.1, by Lemma 2.2 and Lemma 2.4, we have the assertion. □

Claim 3.2.2. *We may assume that R_1 is base point free, that is, $R_1 \otimes \mathcal{O}_X \to L$ is surjective.*

Let $\mathcal{I}$ be the ideal sheaf of X given by

$$\mathrm{Image}(R_1 \otimes \mathcal{O}_X \to L) = \mathcal{I} \cdot L.$$

Let $\mu : X' \to X$ be the blowing-up with respect to $\mathcal{I}$. Then $\mathcal{I} \cdot \mathcal{O}_{X'}$ is invertible. Let t be the canonical section of of $(\mathcal{I} \cdot \mathcal{O}_{X'})^{-1}$, that is, $\mathcal{O}_{X'}(-\mathrm{div}(t)) = \mathcal{I} \cdot \mathcal{O}_{X'}$, and let $L' = \mathcal{I} \cdot \mu^*(L)$. Then, as $\langle (R_1)^n \rangle_{R_0} = R_n$, for $s \in R_n$,

$$\tilde{s} := \mu^*(s) \otimes t^{-n} \in H^0(X', nL').$$

It is easy to see the following properties:

$$\begin{cases} \widetilde{s_1 + s_2} = \widetilde{s_1} + \widetilde{s_2},\ \widetilde{as} = a\tilde{s} & (s_1, s_2, s \in R_n,\ a \in \mathbb{Z}), \\ \widetilde{s_1 \cdot s_2} = \widetilde{s_1} \cdot \widetilde{s_2} & (s_1 \in R_n,\ s_2 \in R_{n'}). \end{cases}$$

Let $\beta_n : R_n \to H^0(X', nL')$ be the homomorphism given by $\beta_n(s) = \tilde{s}$, and $R'_n = \beta_n(R_n)$. Then, by the above properties,

$$\bigoplus_{n=0}^{\infty} \beta_n : \bigoplus_{n=0}^{\infty} R_n \to \bigoplus_{n=0}^{\infty} R'_n$$

yields a ring isomorphism. Let $\|\cdot\|'_n$ be the norm of $(R'_n)_{\mathbb{R}}$ given by $\|\beta_n(s)\|'_n = \|s\|_n$ for $s \in (R_n)_{\mathbb{R}}$. Then

$$\begin{aligned}\|\beta_n(s)\beta_{n'}(s')\|'_{n+n'} = \|\beta_{n+n'}(ss')\|'_{n+n'} = \|ss'\|_{n+n'} \\ \leq \|s\|_n\|s'\|_{n'} = \|\beta_n(s)\|'_n\|\beta_{n'}(s')\|'_{n'}\end{aligned}$$

for all $s \in (R_n)_{\mathbb{R}}$ and $s' \in (R_{n'})_{\mathbb{R}}$. Thus $\bigoplus_{n=0}^{\infty} \beta$ extends to a ring isometry

$$\bigoplus_{n=0}^{\infty}(R_n, \|\cdot\|_n) \xrightarrow{\sim} \bigoplus_{n=0}^{\infty}(R'_n, \|\cdot\|'_n)$$

as normed graded rings over $\mathbb{Z}$. Note that $R'_1 \otimes \mathcal{O}_{X'} \to L'$ is surjective. Hence the claim follows. $\square$

Claim 3.2.3. *We may assume that L is very ample and $R_n = H^0(X, nL)$ for $n \gg 1$.*

By Claim 3.2.2,

$$\left(\bigoplus_{n=0}^{\infty} R_n\right) \otimes \mathcal{O}_X \to \bigoplus_{n=0}^{\infty} nL$$

is surjective, which gives rise to a morphism

$$\phi : X \to Z := \operatorname{Proj}\left(\bigoplus_{n=0}^{\infty} R_n\right)$$

such that $\phi^*(\mathcal{O}_Z(1)) = L$. Note that Z is a projective arithmetic variety. Moreover, there is a natural injective homomorphism $\alpha_n : R_n \to H^0(Z, \mathcal{O}_Z(n))$ such that $\phi_n^*(\alpha_n(s)) = s$ for all $s \in R_n$, where ϕ_n^* is the natural homomorphism $H^0(Z, \mathcal{O}_Z(n))) \to H^0(X, nL)$. If we set $R''_n = \alpha_n(R_n)$, then $\bigoplus_{n=0}^{\infty} \alpha_n$ yields to a ring isomorphism

$$\bigoplus_{n=0}^{\infty} R_n \xrightarrow{\sim} \bigoplus_{n=0}^{\infty} R''_n,$$

so that, as in Claim 3.2.2, there are norms $\|\cdot\|''_0, \ldots, \|\cdot\|''_n, \ldots$ of $R''_0, \ldots, R''_n, \ldots$ such that

$$\bigoplus_{n=0}^{\infty}(R_n, \|\cdot\|_n) \xrightarrow{\sim} \bigoplus_{n=0}^{\infty}(R''_n, \|\cdot\|''_n)$$

as normed graded rings over $\mathbb{Z}$. Moreover, if we set $s''_i = \alpha_1(s_i)$, then $\phi^*(s''_i) = s_i$. Therefore, $\mathrm{Bs}_{\mathbb{Q}}(s''_1, \ldots, s''_l) = \emptyset$ on $Z_{\mathbb{Q}}$. Further, it is well known that α_n is an isomorphism for $n \gg 1$ (cf. [2, the proof of Theorem 5.19 and Remark 5.19.2 in Chapter II]). Hence the claim follows. $\square$

Gathering the assertions of Claim 3.2.1 and Claim 3.2.3, to prove the corollary, we may assume the following:

(a) $s_1, \ldots, s_l \in R_1$ and $\mathrm{Bs}_{\mathbb{Q}}(s_1, \ldots, s_l) = \emptyset$.
(b) L is very ample.
(c) $R_n = H^0(X, nL)$ for $n \gg 1$.

Let $\upsilon : \Sigma_X \to \mathbb{R}_{>0}$ be the constant map given by

$$\upsilon(Y) = \max\{\|s_1\|_1, \ldots, \|s_l\|_1\}$$

for $Y \in \Sigma_X$. Then $(R, \|\cdot\|)$ and υ satisfy the conditions (1), (2) and (3) of Theorem 3.1. Hence the corollary follows. ■

Corollary 3.3. *Let $\overline{L}$ be a continuous hermitian invertible sheaf on X. If there are a positive integer n_0 and $s_1, \ldots, s_l \in H^0(X, n_0 L)$ such that $\mathrm{Bs}_{\mathbb{Q}}(s_1, \ldots, s_l) = \emptyset$, then there is $B \in \mathbb{R}_{>0}$ such that*

$$\lambda_{\mathbb{Q}}(H^0(X, nL), \|\cdot\|_{\sup}) \leq B n^{d(d-1)/2} \left(\max\{\|s_1\|_{\sup}, \ldots, \|s_l\|_{\sup}\}\right)^{n/n_0}$$

for all $n \geq 1$.

Proof. By Theorem 3.2, it is sufficient to show the following lemma. ∎

Lemma 3.4. *Let X be a projective variety over a field k and L an invertible sheaf on X. If there is a positive integer m such that mL is base point free, then $R = \bigoplus_{n=0}^{\infty} H^0(X, nL)$ is noetherian.*

Proof. Since mL is base point free, there are a projective variety Z, an ample invertible sheaf A on Z and a morphism $\phi : X \to Z$ such that $\phi^*(A) = mL$. As A is ample, it is well known that if F is a coherent sheaf on Z, then $R' = \bigoplus_{l=0}^{\infty} H^0(Z, lA)$ is noetherian and $\bigoplus_{l=0}^{\infty} H^0(Z, lA \otimes F)$ is a finitely generated R'-module. Note that

$$\begin{aligned} R = \bigoplus_{n=0}^{\infty} H^0(X, nL) &= \bigoplus_{r=0}^{m-1} \left(\bigoplus_{l=0}^{\infty} H^0(X, (lm + r)L) \right) \\ &= \bigoplus_{r=0}^{m-1} \left(\bigoplus_{l=0}^{\infty} H^0(Z, lA \otimes \phi_*(rL)) \right). \end{aligned}$$

Therefore R is noetherian because R is a finitely generated R'-module. ∎

Remark 3.5. Theorem A and Corollary B in the introduction are consequences of Theorem 3.2 and Corollary 3.3 respectively together with Lemma 1.2. The following examples show that base point freeness by strictly small sections is substantially crucial.

Example 3.6. Let $\mathbb{P}^1_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[X, Y])$ be the projective line over $\mathbb{Z}$ and $\mathcal{O}(1)$ the tautological invertible sheaf on $\mathbb{P}^1_{\mathbb{Z}}$. Then $H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(d))$ is naturally identified with $\mathbb{Z}[X, Y]_d$. Let $\beta, \gamma \in (0, 1)(= \{x \in \mathbb{R} \mid 0 < x < 1\})$ and $\alpha := \beta^{1-(1/\gamma)} > 1$. For each $d \geq 0$, we give a continuous metric $|\cdot|_d$ of $\mathcal{O}(d)$ as follows: for $(x : y) \in \mathbb{P}^1_{\mathbb{Z}}(\mathbb{C})$ and $s \in \mathbb{C}[X, Y]_d$,

$$|s|_d(x : y) = \frac{|s(x, y)|}{\left(\max\{\alpha|x|, \beta|y|\}\right)^d}.$$

We set $\overline{\mathcal{O}}(d) = (\mathcal{O}(d), |\cdot|_d)$. Note that $\overline{\mathcal{O}}(d) = \overline{\mathcal{O}}(1)^{\otimes d}$. Here we have the following:

$$\left\langle \{s \in H^0(X, \mathcal{O}(d)) \mid \|s\|_{\sup} < 1\} \right\rangle_{\mathbb{Z}} = \bigoplus_{d\gamma < i \leq d} \mathbb{Z} X^i Y^{d-i}, \tag{3.6.1}$$

$$\left\langle \{s \in H^0(X, \mathcal{O}(d)) \mid \|s\|_{\sup} \leq 1\} \right\rangle_{\mathbb{Z}} = \bigoplus_{d\gamma \leq i \leq d} \mathbb{Z} X^i Y^{d-i}. \tag{3.6.2}$$

Proof. Indeed, by a straightforward calculation,

$$\|X^i Y^{d-i}\|_{\sup} = \frac{1}{\alpha^i \beta^{d-i}}$$

for $0 \leq i \leq d$. Thus X^iY^{d-i} is a strictly small section for i with $d\gamma < i \leq d$ because $\alpha^i\beta^{d-i} > 1$. On the other hand, for $s = \sum_{i=0}^{d} a_iX^iY^{d-i} \in \mathbb{C}[X,Y]_d$, we can see

$$
\begin{aligned}
\|s\|_{\sup} &\geq \sup\left\{|s|_d(z:1) \mid |z| = \frac{\beta}{\alpha}\right\} = \frac{1}{\beta^d}\sup\left\{|s(z,1)| \mid |z| = \frac{\beta}{\alpha}\right\} \\
&\geq \frac{1}{\beta^d}\sqrt{\int_0^1 \left|s\left(\left(\frac{\beta}{\alpha}\right)e^{2\pi\sqrt{-1}\theta}, 1\right)\right|^2 d\theta} \\
&= \frac{1}{\beta^d}\sqrt{\sum_{0\leq i,j\leq d}\int_0^1 a_i\bar{a}_j\left(\frac{\beta}{\alpha}\right)^{i+j} e^{2\pi\sqrt{-1}(i-j)\theta}d\theta} \\
&= \sqrt{\sum_{i=0}^{d}\left(\frac{|a_i|}{\alpha^i\beta^{d-i}}\right)^2}.
\end{aligned}
$$

Thus, if $s = \sum_{i=0}^{d} a_iX^iY^{d-i} \in \mathbb{Z}[X,Y]_d$ is a strictly small section, then $a_j = 0$ for j with $0 \leq j \leq d\gamma$ because $\alpha^j\beta^{d-j} \leq 1$. These observations yield (3.6.1). Similarly we obtain (3.6.2). ■

Example 3.7. Let $\mathbb{P}^2_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[X,Y,Z])$ be the projective plane over $\mathbb{Z}$ and $\mathcal{O}(1)$ the tautological invertible sheaf on $\mathbb{P}^2_{\mathbb{Z}}$. Let Δ be the arithmetic subvariety of $\mathbb{P}^2_{\mathbb{Z}}$ given by the homogeneous ideal $Y\mathbb{Z}[X,Y,Z] + Z\mathbb{Z}[X,Y,Z]$. Let $\mu : X \to \mathbb{P}^2_{\mathbb{Z}}$ be the blowing-up along Δ and E the exceptional divisor of μ. Note that E is a Cartier divisor. We set $L = \mu^*(\mathcal{O}(1)) + \mathcal{O}_X(E)$ and $R = \bigoplus_{n=0}^{\infty} H^0(X, nL)$. Since $\mu_*(nL) = \mathcal{O}(n)$ for all $n \in \mathbb{Z}_{\geq 0}$, the natural ring homomorphism

$$
\mu^* : \bigoplus_{n=0}^{\infty} H^0(\mathbb{P}^2_{\mathbb{Z}}, \mathcal{O}(n)) \longrightarrow R
$$

yields a ring isomorphism, and

$$
\{x \in X \mid s(x) = 0 \text{ for all } s \in H^0(X, nL)\} = E
$$

for $n \in \mathbb{Z}_{>0}$. Here we give a metric $|\cdot|_{FS}$ of $\mathcal{O}(1)$ in the following way: for $s \in H^0(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}(1)) = \mathbb{C}[X,Y,Z]_1$ and $(x:y:z) \in \mathbb{P}^2(\mathbb{C})$,

$$
|s|_{FS}(x:y:z) = \frac{|s(x,y,z)|}{\sqrt{|x|^2+|y|^2+|z|^2}}.
$$

We set $\overline{\mathcal{O}}(n) = (\overline{\mathcal{O}}(1), |\cdot|_{FS})^{\otimes n}$. Then it is easy to check that $\|X^iY^jZ^k\|_{\sup} \leq 1$ for all $n > 0$ and $i,j,k \in \mathbb{Z}_{\geq 0}$ with $i+j+k = n$. Let t be the canonical section of $\mathcal{O}_X(E)$. We choose a C^∞-metric $|\cdot|_E$ of $\mathcal{O}_X(E)$ such that $\|t\|_{\sup} < 1$, and set

$$
\overline{L} = \mu^*(\overline{\mathcal{O}}(1)) + (\mathcal{O}_X(E), |\cdot|_E).
$$

Then $\|\mu^*(X^iY^jZ^k) \otimes t\|_{\sup} < 1$ for all $n > 0$ and $i,j,k \in \mathbb{Z}_{\geq 0}$ with $i+j+k = n$. As a consequence, R_n has non-empty base loci, but possesses a free basis consisting of strictly small sections. However, in this example, the free basis comes from the base point free $\mathbb{Z}$-module $H^0(\mathbb{P}^2_{\mathbb{Z}}, \mathcal{O}(n))$.

4. VARIANTS OF ARITHMETIC NAKAI-MOISHEZON'S CRITERION

Let X be a projective arithmetic variety and Y an arithmetic subvariety of X. Let $\overline{L}$ be a continuous hermitian invertible sheaf on X. We denote

$$\operatorname{Image}(H^0(X,L) \to H^0(Y, L|_Y))$$

by $H^0(X|Y,L)$. Let $\|\cdot\|^{X|Y}_{\mathrm{sup,quot}}$ be the the quotient norm of $H^0(X|Y,L)\otimes_{\mathbb{Z}}\mathbb{R}$ induced by

$$H^0(X,L)\otimes_{\mathbb{Z}}\mathbb{R} \twoheadrightarrow H^0(X|Y,L)\otimes_{\mathbb{Z}}\mathbb{R}$$

and the norm $\|\cdot\|_{\sup}$ on $H^0(X,L)\otimes_{\mathbb{Z}}\mathbb{R}$. As in [4], we define $\widehat{\mathrm{vol}}_{\mathrm{quot}}(X|Y,\overline{L})$ to be

$$\widehat{\mathrm{vol}}_{\mathrm{quot}}(X|Y,\overline{L}) := \limsup_{m\to\infty} \frac{\log \#\left\{ s \in H^0(X|Y,mL) \mid \|s\|^{X|Y}_{\mathrm{sup,quot}} \leq 1 \right\}}{m^{\dim Y}/(\dim Y)!}.$$

Then we have the following variants of arithmetic Nakai-Moishezon's criterion. Theorem 4.2 is a slight generalization of the original criterion due to Zhang [7], that is, we do not assume that $L_{\mathbb{Q}}$ is ample.

Theorem 4.1. *If $\widehat{\mathrm{vol}}_{\mathrm{quot}}(X|Y,\overline{L}) > 0$ for all arithmetic subvarieties Y of X, then $L_{\mathbb{Q}}$ is ample and there is a positive integer n_0 such that, for all $n \geq n_0$, $H^0(X,nL)$ has a free $\mathbb{Z}$-basis consisting of strictly small sections.*

Proof. First of all, note that

$$\widehat{\mathrm{vol}}\left(Y, \overline{L}\big|_Y\right) \geq \widehat{\mathrm{vol}}_{\mathrm{quot}}(X|Y,\overline{L}) > 0$$

for all arithmetic subvarieties Y of X. In particular, by [5, Corollary 2.4] or [3, Theorem 4.6], $L_{\mathbb{Q}}|_{Y_{\mathbb{Q}}}$ is big. Thus, by algebraic Nakai-Moishezon's criterion, $L_{\mathbb{Q}}$ is ample. Let us consider a normed graded ring

$$(R, \|\cdot\|) = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} (H^0(X,nL), \|\cdot\|_{\sup}).$$

As $L_{\mathbb{Q}}$ is ample, R satisfies the conditions (1) and (2) of Theorem 3.1. Moreover, if we take a sufficiently small positive number ϵ, then

$$\widehat{\mathrm{vol}}_{\mathrm{quot}}(X|Y,\overline{L} - \overline{\mathcal{O}}(\epsilon)) > 0$$

by [4, (2) in Proposition 6.1], which means that we can choose a map $\upsilon : \Sigma_X \to \mathbb{R}_{>0}$ such that $\upsilon(Y) < 1$ for all $Y \in \Sigma_X$ and the condition (3) of Theorem 3.1 holds for $(R,\|\cdot\|)$ and υ. Thus the last assertion follows. ∎

Theorem 4.2. *We assume that X is generically smooth, the metric of $\overline{L}$ is C^∞, L is nef on every fiber of $X \to \operatorname{Spec}(\mathbb{Z})$ and that the first Chern form $c_1(\overline{L})$ is semipositive on $X(\mathbb{C})$. If $\widehat{\deg}\left((\hat{c}_1(\overline{L})\big|_Y)^{\dim Y}\right) > 0$ for all arithmetic subvarieties Y of X, then $L_{\mathbb{Q}}$ is ample and there is a positive integer n_0 such that, for all $n \geq n_0$, $H^0(X,nL)$ has a free $\mathbb{Z}$-basis consisting of strictly small sections.*

Proof. By virtue of the Generalized Hodge index theorem [3, Theorem 6.2],

$$\widehat{\mathrm{vol}}\left(Y, \overline{L}\big|_Y\right) \geq \widehat{\deg}\left((\hat{c}_1(\overline{L})\big|_Y)^{\dim Y}\right) > 0.$$

Thus $L_{\mathbb{Q}}$ is ample as in the proof of Theorem 4.1. In particular, if we set

$$(R, \|\cdot\|) = \bigoplus_{n\geq 0} (H^0(X,nL), \|\cdot\|_{\sup}),$$

then R satisfies the conditions (1) and (2) of Theorem 3.1. As $\widehat{\mathrm{vol}}\left(Y, \overline{L}\big|_Y\right) > 0$, we can find a non-zero strictly small section s of $n_1 L|_Y$ for some positive integer n_1. By [7, Theorem 3.3 and Theorem 3.5], there are a positive integer n_2 and $s' \in H^0(X, n_2 n_1 L) \otimes \mathbb{R}$ with $s'|_Y = s^{\otimes n_2}$ and $\|s'\|_{\sup} < 1$. Thus a map $\upsilon : \Sigma_X \to \mathbb{R}_{>0}$ given by

$$\upsilon(Y) = \left(\|s'\|_{\sup}\right)^{1/n_1 n_2}$$

satisfies the condition (3) of Theorem 3.1. Hence the theorem follows from Theorem 3.1. ∎

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